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Two-parametric solution to graded Yang–Baxter equation and two-parametric $U_{uv}gl(1|1)$ algebra as a Hopf algebra

H Yan†, Y Zhou‡ and T-H Zhu§

† China Center of Advanced Science and Technology (World Laboratory), PO Box 8730, Beijing 100080, People's Republic of China
and
Institute of Theoretical Physics, Academia Sinica, PO Box 2735, Beijing 100080, People's Republic of China

‡ Department of Physics, Kansas State University, Manhattan, KS 66506, USA

§ Department of Physics, Columbia University, New York, NY 10027, USA

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Abstract. We discuss a two-parametric solution of the graded Yang–Baxter equation (YBE) and perform the Yang–Baxterization to obtain the solution to quantum YBE. In the formalism developed in [1–4], we give the two-parametric quantized superalgebra $U_{uv}gl(1|1)$ and prove that this algebra is a Hopf algebra with the Hopf operations explicitly provided.

1. Introduction

Quantum groups and trigonometric quantum Yang–Baxter equations [1-8] are found to be closely related with the physical theories of integrable models, inverse scattering method for nonlinear evolution equation, factorizable S-matrix and integrable field theory, conformal field theories and topological field theory and Chern–Simons theory. On the one hand, this mathematical theory has also been generalized to include the supersymmetric case [6], and on the other, there is currently much interest in the generalizations of multi-parametric solutions of Yang–Baxter equations [9–12] and multi-parametric deformation of the Lie algebras [9–12]. In this paper, we will concentrate on the multi-parametric solution of graded YBE and multi-parametric deformation of graded Lie algebras [7, 8].

To set up notations, we recall some well known facts. The quantum Yang–Baxter equation [13] reads

$$R_{12}(\lambda)R_{13}(\lambda\mu)R_{23}(\mu) = R_{23}(\mu)R_{13}(\lambda\mu)R_{12}(\lambda) \tag{1}$$

where $R_{ij}(x) \in \mathrm{End}_{\mathbf{C}}(V\otimes V\otimes V)$ is a matrix acting on the ith and jth spaces non-trivially and trivially on the third one, with $x \in \mathbf{C}$ the spectral parameter. The YBE takes different forms in literature, and besides the form in (1), we have equivalently

$$\check{R}_{12}(\lambda)\check{R}_{23}(\lambda\mu)\check{R}_{12}(\mu) = \check{R}_{23}(\mu)\check{R}_{12}(\lambda\mu)\check{R}_{23}(\lambda) \tag{2}$$

where it should be noted that $\check{R}(x) = PR(x)$ and P denotes the permutation matrix in $V\otimes V$. The second form of YBE is valid to the supersymmetric case, while the first

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one will have to be modified when we deal with graded (Lie or quantum) algebras [6] (we will come to this point again later).

Let $S_j \in \mathrm{End}(V\otimes^{n-1})$ be given by

$$S_j = I^{(1)} \otimes I^{(2)} \otimes \cdots \otimes I^{(i-1)} \otimes \check{R} \otimes I^{(i+1)} \otimes \cdots \otimes I^{(n-1)} \tag{3}$$

where $\check{R}$ is spectrum-independent solution of (2); then each of such solutions leads to an n-dimensional braid-group representation (BGR), i.e.

$$\begin{aligned} &S_i S_{i+1} S_i = S_{i+1} S_i S_{i+1} \\ &S_i S_j = S_j S_i \qquad \text{for} \quad |i-j| \geqslant 2 . \end{aligned} \tag{4}$$

As various solutions of BGRs are easily found, the theory of Yang–Baxterization is often applied to obtain the solutions of quantum (or spectrum-dependent) YBE. The BGRs are usually obtained from the trigonometric or hyperbolic solutions of YBE by setting the spectral parameter to infinity. The standard method in obtaining BGR from the universal R-matrix for the q-deformation of Lie algebras gives a series of BGRs called *standard*, which revert to the permutation matrix when the deformation parameter $q \to 1$. Other types of solutions of YBE are usually called *non-standard* ones [14, 20].

According to [19], the solutions of YBE (2), which revert to the superpermutation matrix when $q \to 1$ are nothing but the solutions that correspond to the graded algebras. For space $V \otimes V$ with V being 2D linear space, the superpermutation matrix reads

$$\mathcal{P}_{12} = \eta P_{12} = \begin{bmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & -1 \end{bmatrix} . \tag{5}$$

The YBE in (1) should be modified to the following form:

$$(\eta_{12} R_{12})(\eta_{13} R_{13})(\eta_{23} R_{23}) = (\eta_{23} R_{23})(\eta_{13} R_{13})(\eta_{12} R_{12}) \tag{6}$$

to be valid to the supersymmetric case, where $\eta = \mathrm{diag}(1,1,1,-1)$ is a super phase factor, while (2) remains correct. Therefore although in [17] ten solutions are obtained for (1), some of which are non-standard ones, super-solutions are obviously not included.

In section 2, we will supply a multi-parametric solution of BGR, and perform the Baxterization to obtain a solution to the graded quantum YBE. A general method of inserting the spectral parameter into the graded R-matrix is suggested. In section 3, we give the multi-parametric graded quantum super algebra $U_{uv}gl(1|1)$, and the graded quantum Yang–Baxter equation, and though the main results in this section are included in [7, 8]†, we go further to show that this algebra is a Hopf algebra with provided Hopf operations‡.

† We thank the referee for pointing out this fact.

‡ We realized this point after the referee's important suggestion.

2. A solution to graded YBE and Baxterization

It can be easily verified that the following is a solution to the YBE (2) or (6):

$$\check{R} = \begin{bmatrix} q & & & \\ & 0 & u & \\ & v & q - uv/q & \\ & & & -uv/q \end{bmatrix} . \tag{7}$$

This solution is apparently three-parameter-dependent, but one of them can be eliminated by a proper rescaling. Therefore

$$\check{R} = qE_{11} \otimes E_{11} + uE_{21} \otimes E_{12} + vE_{12} \otimes E_{21} + \left(q - \frac{uv}{q}\right) E_{11} \otimes E_{22} - \frac{uv}{q} E_{22} \otimes E_{22} \tag{8}$$

where $E_{\alpha\beta}$ are 2×2 matrices and

$$\left(E_{\alpha\beta}\right)_{ij} = \delta_{\alpha i}\delta_{\beta j} . \tag{9}$$

Following the procedure of Baxterization, we can put a spectral parameter into the above matrix such that it becomes a solution of the quantum YBE. We can always diagonalize S and rewrite it as

$$\check{R} = \sum_{i=1}^{2} \Lambda_i P_i \tag{10}$$

where $\Lambda_1 = q$ and $\Lambda_2 = -uv/q$ are distinct eigenvalues of $\check{R}$, i.e.

$$(\check{R} - q)\left(\check{R} + \frac{uv}{q}\right) = 0 \tag{11}$$

and P_i are projectors such that

$$P_i P_j = \delta_{ij} P_j \qquad \text{and} \qquad \sum_{i=1}^{2} P_i = I . \tag{12}$$

The trigonometric solution of the quantum YBE (2) satisfies

$$\check{R}(x) = \sum_{i=1}^{2} \rho_i(x) P_i \tag{13}$$

where the unknown factors can actually be given by the following formulas:

$$\rho_1(x) = \left(x + \frac{\lambda_1}{\lambda_2}\right) \qquad \text{and} \qquad \rho_2(x) = \left(1 + x\frac{\lambda_1}{\lambda_2}\right) \tag{14}$$

where λ_1 and λ_2 are permutations of the eigenvalues Λ_1 and Λ_2, if we require that the quantum solution satisfies boundary, initial and unitarity conditions as follows:

$$\lim_{x \to 0} \check{R}(x) = cS \qquad \check{R}(1) = \text{const} \times I \qquad \check{R}(x)\check{R}(x^{-1}) = \rho(x) I . \tag{15}$$

The formula in (14) is actually the same as that developed in [3] and [14] for standard solutions, and we expect it to be true in more general cases [15] where the number of eigenvalues is $n \geqslant 2$, i.e.

$$\check{R}(x) = \sum_{i=1}^{n} \rho_i(x) P_i \tag{16}$$

where

$$\rho_i(x) = \left(1 + x\frac{\lambda_1}{\lambda_2}\right)\left(1 + x\frac{\lambda_2}{\lambda_3}\right)\cdots\left(1 + x\frac{\lambda_{i-1}}{\lambda_i}\right)\left(x + \frac{\lambda_i}{\lambda_{i+1}}\right)\cdots\left(x + \frac{\lambda_{n-1}}{\lambda_n}\right). \tag{17}$$

The present case is the easiest, in that there are only two eigenvalues. The final result reads

$$\check{R}(x) = \begin{bmatrix} xq - x^{-1}uv/q & & & \\ & x^{-1}(q - uv/q) & (x - x^{-1})u & \\ & (x - x^{-1})v & (q - uv/q) & \\ & & & x^{-1}q - xuv/q \end{bmatrix}. \tag{18}$$

This solution can be further generalized to insert more parameters,

$$\check{R}(x) = \begin{bmatrix} xq - x^{-1}uv/q & & & \\ & x^{-k}(q - uv/q) & u^{1-c_1}v^{c_2}(x - x^{-1}) & \\ & u^{c_1}v^{1-c_2}(x - x^{-1}) & x^{k}(q - uv/q) & \\ & & & x^{-1}q - xuv/q \end{bmatrix} \tag{19}$$

where q, u and v are deformation parameters, x is spectral parameter, and k is *gauge* tranformation constant [16], while c_1 and c_2 appear as the pure effect of multi-parametric deformation. When $k = 1$ and $c_1 = c_2 = 0$, we arrive at the solution (18) obtained by standard Baxterization.

3. Two-parametric graded quantum algebra $U_{uv}gl(1|1)$

Employing the method developed by Faddeev, Reshetikhin, Takhtajan, Kulish and Sklyanin and others [1–4], one can obtain from a solution of YBE the quantum algebra, equipped automatically with Hopf operations. This method can be generalized to the supersymmetric case [18, 19].

As was mentioned in section 1, the graded YBE is (6), and the Yang–Baxter algebra (YBA) is rewritten

$$R_{12}T_1\,(\eta_{12}T_2\eta_{12}) = (\eta_{12}T_2\eta_{12})\,T_1R_{12} \tag{20}$$

where

$$T_1 = T \otimes I \qquad T_2 = I \otimes T\,. \tag{21}$$

The co-associativity of the triple product of T_1, $\eta_{12}T_2\eta_{12}$ and $\eta_{23}\eta_{13}T_3\eta_{13}\eta_{23}$ in space $V \otimes V \otimes V$ yields the YBE (6). The dual algebra relation is

$$R_{21}L_1^{(\epsilon)}\left(\eta_{12}L_2^{(\epsilon')}\eta_{12}\right) = \left(\eta_{12}L_2^{(\epsilon')}\eta_{12}\right)L_1^{(\epsilon)}R_{21} \tag{22}$$

where ϵ or ϵ' takes + or −. The co-associativity condition also guarantees the pairing condition of the dual YBAs, i.e.

$$\langle L_1^{(\pm)}, T_2 \rangle = R_{12}^{(\pm)} \tag{23}$$

and

$$R_{12}^{(+)} = \eta_{12} R_{21} \eta_{12} \qquad R_{12}^{(-)} = R_{12}^{-1} \tag{24}$$

and the matrix R has already been assumed non-singular. The dual algebra relation (22) can be rewritten

$$\check{R}_{12}\left(\eta_{12} L_1^{(\epsilon)} \eta_{12}\right) L_2^{(\epsilon')} = \left(\eta_{12} L_1^{(\epsilon')} \eta_{12}\right) L_2^{(\epsilon)} \check{R}_{12} \tag{25}$$

where we have applied the following relations:

$$\eta_{ab} = \eta_{ba} \qquad \eta_{ab}\eta_{cd} = \eta_{cd}\eta_{ab} \qquad \eta_{ac}\eta_{bc}R_{ab} = R_{ab}\eta_{ac}\eta_{bc}\,. \tag{26}$$

The first two equations above are identities about the super phase factors, and the third one is the super version of the weight conservation [19].

Now we are in the position to consider the R-matrix of the specific form in (7) and write $L^{(\pm)}$ as upper and lower triangular matrices as follows:

$$L^{(+)} = \begin{bmatrix} k^- & (q - uv/q)x \\ 0 & l^+ \end{bmatrix} \qquad L^{(-)} = \begin{bmatrix} k^+ & 0 \\ (q - uv/q)y & l^- \end{bmatrix}. \tag{27}$$

From (25) we have the algebraic relations for the algebra spanned by the elements $x, y, k^\epsilon, l^\epsilon$,

$$\begin{aligned} &l^\epsilon k^{\epsilon'} = k^{\epsilon'} l^\epsilon \qquad (\epsilon\ \epsilon' = +, -) \qquad k^+ x k^- = q v^{-1} x \\ &l^+ x l^- = q^{-1} v x \qquad k^+ y k^- = q^{-1} u y \qquad l^+ y l^- = q u^{-1} y \\ &x^2 = y^2 = 0 \qquad u y x + v x y = \frac{k^+ l^+ - k^- l^-}{q - uv/q} \end{aligned} \tag{28}$$

and $k^+k^- = k^-k^+$ and $l^+l^- = l^-l^+$ are in the centre of the algebra (denoted $U_{uv}gl(1|1)$), i.e.

$$[k^\pm k^\mp, \bullet\,] = 0 \qquad [l^\pm l^\mp, \bullet\,] = 0 \qquad \forall \bullet \in\ U_{uv}gl(1|1)\,. \tag{29}$$

The relations in (29) allow one to set

$$k^- = (k)^{-1} \qquad k^+ = k \qquad l^{=}(l)^{-1} \qquad l^+ = l \tag{30}$$

and therefore $U_{uv}gl(1|1) = \mathrm{span}\{1, k, l, x, y\}$. When $u/v \to 1$, the single-parametric deformed quantum algebra $U_q gl(1|1)$ is recovered, and if we further set $q \to 1$ then the Lie universal enveloping superalgebra $U gl(1|1)$ is recovered.

The co-product of this algebra can be determined by

$$\Delta(L^{(\pm)}) = L^{(\pm)} \dot{\otimes} L^{(\pm)} \tag{31}$$

where $\overset{\bullet}{\otimes}$ denotes the tensor product combined with the usual matrix multiplication

$$\begin{aligned} &\Delta\,(k^{\pm}) = k^{\pm}\otimes k^{\pm} \qquad \Delta\,(l^{\pm}) = l^{\pm}\otimes l^{\pm} \\ &\Delta\,(x) = x\otimes k + l^{-1}\otimes x \qquad \Delta\,(y) = y\otimes l + k^{-1}\otimes y \end{aligned} \tag{32}$$

which is an operation of algebra homomorphism, i.e. $\forall a,b\in U_{uv}gl(1|1)$, $\Delta(ab) = \Delta(a)\Delta(b)$. We can also give another homomorphism, co-unit denoted ϵ, and an antihomomorphism, antipodal mapping denoted S, as follows:

$$\begin{aligned} &\epsilon\,(x) = \epsilon\,(y) = 0 \qquad \epsilon\,(k) = \epsilon\,(l) = 1 \\ &S\,(x) = -lxk^{-1} \qquad S\,(y) = -kyl^{-1} \\ &S\,(k) = k^{-1} \qquad S\,(l) = l^{-1} \end{aligned} \tag{33}$$

where the antihomomorphism of the operation of antipodal mapping means that for any $a,b\in U_{uv}gl(1|1)$, $S(ab) = -S(b)S(a)$. Note that a minus sign appears because of the super nature of this algebra, which makes it differ with the well known quantum algebras. The consistency of above-defined operations with the algebraic relations can be easily checked. Therefore the two-parametric deformed algebra is a Hopf algebra, by definition. It has been well known that the two-parametric quantized algebra of $sl(2)$ is not a Hopf algebra[9], in that it does not have consistent definitions of co-unit and antipodal mapping. However, the newly defined two-parametric deformation of $gl(1|1)$ is a Hopf algebra, and this may be interesting.

To end this paper, we want to give the quantum version of the Yang–Baxter relation (25)

$$\check{R}_{12}(\lambda\mu^{-1})\,(\eta_{12}L_1(\lambda)\eta_{12})\,L_2(\mu) = (\eta_{12}L_1(\mu)\eta_{12})\,L_2(\lambda)\check{R}(\lambda\mu^{-1})_{12} \tag{34}$$

where

$$L_1(\lambda) = L(\lambda)\otimes 1 \qquad L_2(\mu) = 1\otimes L(\mu) \tag{35}$$

and

$$L(\lambda) = \begin{bmatrix} \lambda k - \lambda^{-1}k^{-1} & (q-uv/q)y \\ (q-uv/q)x & \lambda l^{-1} - \lambda^{-1}l \end{bmatrix}$$

$$\check{R}_{12}(\lambda) = \begin{bmatrix} \lambda q - \lambda^{-1}uv/q & & & \\ & q-uv/q & u(\lambda-\lambda^{-1}) & \\ & v(\lambda-\lambda^{-1}) & q-uv/q & \\ & & & \lambda^{-1}q - \lambda uv/q \end{bmatrix} \tag{36}$$

which can be obtained from (19) by setting $k = c_1 = c_2 = 0$. This quantized form may be useful if one is concerned with relating this solution of the Yang–Baxter equation and possibly the newly defined Hopf algebra with a quantum spin model.

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